Aristodesmus, which suggests this link, is at present placed in the Procolophonia, a group separated from its recent association with *Pareiasaurus* and restored to its original independence, because it has two occipital condyles, with the occipital plate vertical, and without lateral vacuities, and has the shoulder girdle distinct from Pareiasauria in the separate pre-coracoid extending in advance of the scapula.

III. "Octonions." By Alex. McAulay, M.A., Lecturer in Mathematics and Physics, University of Tasmania. Communicated by Rev. N. M. Ferrers, D.D., F.R.S. Received November 28, 1895.

(Abstract.)

Octonions is a name adopted for various reasons in place of Clifford's *Bi-quaternions*.

Formal quaternions are symbols which *formally* obey all the laws of the quaternion *symbols*, q (quaternion), x (scalar), ρ (vector) ϕ (linear function in both its ordinary meanings), ϕ' (conjugate of ϕ), i, j, k, ζ, Kq, Sq, Tq, Uq, Vq. Octonions are in this sense formal quaternions. Each octonion symbol, however, requires for its specification just double the number of scalars required for the corresponding quaternion symbol. Thus, of every quaternion formula involving the above symbols there is a geometrical interpretation more general than the ordinary quaternion one, an octonion interpretation. The new interpretation, like the old, treats space impartially, *i.e.*, it has no special reference to an arbitrarily chosen origin or system of axes.

If Q is an octonion and q a quaternion, the symbols, which in octonions correspond to Kq, Sq, Tq, Uq, Vq in quaternions, are denoted by KQ, SQ, TQ, UQ, MQ. KQ is called the conjugate of Q, SQ the scalar octonion part, TQ the augmenter, UQ the twister, and MQ the motor part.

If A is a "motor" whose axis intersects the axis of Q perpendicularly, QA is also a motor intersecting Q perpendicularly. UQ.A is obtained from A by a combined translation along and rotation about the axis of Q; *i.e.*, by a "twist" about the axis of Q. TQ.A is obtained from A by increasing the "rotor" part of A in a definite *ratio*, and by increasing the "pitch" of A by a definite *addition*. UQ $=$ U_1Q U_2Q, where when Q is thus regarded as an operator, U_1Q is a "versor," *i.e.*, it effects the rotation mentioned; and U_2Q is a "translator," *i.e.*, it effects the translation mentioned. Similarly, TQ $=$ T_1Q.T_2Q where T_1Q is a "tensor," *i.e.*, it effects the ratio

increase mentioned; and T_2Q is an "additor," *i.e.*, it effects the addition-increase mentioned. The amount by which the pitch of the motor operand A is increased is called the pitch of the octonion operator Q. When a motor B is itself thus considered as an operator its versor is a quadrantal versor and its translator is unity, *i.e.*, it does not translate the operand at all; hence the twister of a motor is a quadrantal versor. The tensor of B is the magnitude of its rotor part, and the amount by which it increases the pitch of the operand is its own pitch.

These results are all established by aid of quaternions on a purely Euclidean basis. They have, of course, mechanical interpretations in connection with (1) the instantaneous motion of a rigid body, (2) a system of forces, (3) the momentum of a system of moving matter, and (4) a system of impulses. The corresponding motors are called velocity motors (Sir Robert Ball's "twist on a screw"), force motors ("wrench on a screw"), momentum motors, and impulse motors ("impulsive wrench"). If A and B are two force motors A+B is the force motor of the system of forces obtained by the composition of the two systems corresponding to A and B; and similarly for the motors of the other types.

The octonion operator $Q(\ )Q^{-1}$ is exactly analogous to the quaterion operator $q(\ )q^{-1}$. It displaces the octonion operand in the most general manner as a rigid body; it translates the operand parallel to the axis of Q through a distance double of that through which Q when regarded as a motor operator translates the motor; and it rotates it as a rigid body round the axis of Q through double the angle of Q. If A is the velocity motor of a rigid body which in the time t has suffered the displacement $Q(\ )Q^{-1}$,

$$A = 2MQQ^{-1},$$

and if R is any octonion fixed relative to the body,

$$R = MAMR.$$

The geometrical connections between A, B (two motors), MAB, and SAB are examined. The axis of MAB is the shortest distance between A and B. Its rotor part bears the same relation to the rotor parts of A and B that the vector $V\alpha\beta$ does to the vectors α and β. The pitch of MAB is the sum of $d \cot\theta$ and the pitches of A and B, where d is the distance and θ the angle between A and B, d being reckoned positive or negative, according as the shortest twist which will bring the rotor of either A or B into coincidence, both as to axis and sense, with the rotor of the other is a right-handed or left-handed one.

A scalar octonion such as SAB requires two ordinary scalars, S_1AB and sAB to specify it. S_1AB bears to the rotor parts of A and B the

same relation as the scalar $S\alpha\beta$ bears to the vectors α and β. sAB is most simply described mechanically. If A is the velocity motor of a rigid body on which the force motor B is acting, $-s$AB is the rate at which the corresponding system of forces is doing work on the body. $-s$AB is, therefore, the product of the tensors of A and B, multiplied by what Sir Robert Ball calls the virtual coefficient of the two corresponding screws. If A and B are rotors, sAB is $\pm$ six times the volume of the tetrahedron which has A and B for a pair of opposite edges. $s\mathrm{AB}/\mathrm{S}_1\mathrm{AB}$ is what is called the pitch of the scalar octonion SAB. It is the sum of $-d\tan\theta$ and the pitches of A and B.

If, A, B, C are three motors, $\mathrm{S}_1\mathrm{ABC}$ bears to their rotor parts the same relation that the scalar $S\alpha\beta\gamma$ bears to the three vectors α, β, γ. The pitch of SABC is the sum of $d\cot\theta - e\tan\phi$, and the pitches of A, B, and C; where d and θ are related to A and B, as before, and where e is the distance, and ϕ the angle between C and the shortest distance of A and B.

The rotor part $\mathrm{M}_1\mathrm{ABC}$ of the motor MABC bears, as to direction and magnitude the same relation to the rotor parts of A, B, C as the vector $V\alpha\beta\gamma$ bears to the vectors α, β, γ. The pitch of MABC is the sum of $(e\tan\phi - d\cot\theta)/(\cot^2\theta\tan^2\phi + \cot^2\theta + \tan^2\phi)$ and the pitches of A, B, and C.

Similarly as to the rotor $\mathrm{M}_1.(\mathrm{MAB})\mathrm{C}$ of the motor M(MAB)C. The pitch of M(MAB)C is the sum of $d\cot\theta + e\cot\phi$ and the pitches of A, B, C.

A finite motor, whose pitch is infinite, is called a lator. [The term "vector" is not here used, because though lators and vectors have the same fundamental geometrical properties, and obey the same laws of addition, they do not obey the same laws of multiplication, for the product of two lators is always zero.] Thus every motor consists of a rotor part and a parallel lator. If Q is an octonion, $\mathrm{M}_1\mathrm{Q}$ stands for the rotor of the motor of Q and mQ for a coaxial rotor of the same magnitude and sense as the lator of the motor of Q.

If MAB $= 0$, either A and B are coaxial, or one of them is a lator parallel to the axis of the other, or they are both lators. If $\mathrm{M}_1\mathrm{AB} = 0$, either A and B are parallel, or one of them at least is a lator. If $m\mathrm{AB} = 0$, either $(p+p')\sin\theta + d\cos\theta = 0$ (where d and θ are as before, and p and p' are the pitches of A and B), or one is a lator parallel to the axis of the other, or they are both lators.

If SAB $= 0$, either they intersect perpendicularly, or one is a lator perpendicular to the axis of the other, or they are both lators. If $\mathrm{S}_1\mathrm{AB} = 0$, either they are perpendicular, or one of them at least is a lator. If $s\mathrm{AB} = 0$ either $(p+p')\cos\theta = d\sin\theta$, or one is a lator perpendicular to the axis of the other, or they are both lators.

The necessary and sufficient condition to ensure that SABC $= 0$ is

either two independent motors of the complex A, B, C are lators or XA + YB + ZC = 0, where X, Y, Z are scalar octonions whose ordinary scalars (S, X, &c.) are not all zero.

If B and C have definite not parallel axes, XB + YC is any motor that intersects the shortest distance of B and C perpendicularly, where X and Y are arbitrary scalar octonions.

The analogue in octonions of the linear vector function of a vector in quaternions is called a *commutative* linear motor function of a motor. It is not the most general form of a linear motor function of a motor. The latter is called a *general* function. If a commutative function is such that, acting on an arbitrary rotor through a definite point, it reduces the operand to a rotor through the same point it is called a *pencil* function, and the point is called the *centre* of the pencil function. The geometrical relations between a pencil function and rotors through its centre are precisely the same as the geometrical relations between a linear vector function of a vector and vectors. When a commutative function degenerates into a lator function (for all values of the motor operand) the geometrical relations between the function and rotors in general are also precisely the same as the corresponding quaternion relations. A general function involves thirty-six ordinary scalars, a general self-conjugate twenty-one, a commutative function eighteen, a commutative self-conjugate twelve, a pencil function twelve, and a self-conjugate pencil function nine. [In the case of the pencil function three of the scalars go to specify the centre.] ϖ, a general self-conjugate, is called an *energy* function when $s\mathrm{E}\varpi\mathrm{E}$ is not positive for any motor E; it is a *partial* or *complete* energy function, according as $s\mathrm{E}\varpi\mathrm{E}$ is zero for some values of E or for none.

We here pass over for the most part those properties of the commutative function which are immediately deducible from the fact that octonions are formal quaternions.

Let ϕ for the present stand for a commutative function. ϕ satisfies a cubic with scalar octonion coefficients. This cubic generally has three definite scalar octonion roots; but sometimes it has an infinite number, sometimes only one, and sometimes none at all. The cubic (called the ϕ_1 cubic), whose coefficients are the ordinary scalar parts of the coefficients of the ϕ cubic, has an important bearing on the geometrical properties of ϕ.

ϕ can always be put in a trinomial form analogous to the corresponding quaternion trinomial form.

If X is a root of the ϕ cubic corresponding to a single root of the ϕ_1 cubic, $\phi-\mathrm{X}$ can be put in a binomial form, and there is a line such that if E be any motor coaxial with the line $\phi\mathrm{E} = \mathrm{XE}$, so that $\phi\mathrm{E}$ is coaxial with E. If X corresponds to a repeated root of the ϕ_1 cubic, this is not always true.

If X is root of the ϕ cubic that cubic can always be put in the form $(\phi-\mathrm{X})(\phi^2-\mathrm{N}'\phi+\mathrm{N}) = 0$, even when there is no second root. If X corresponds to a single root of the ϕ_1 cubic $(\phi^2-\mathrm{N}'\phi+\mathrm{N})\mathrm{E} = 0$, if E is any motor which intersects a certain line perpendicularly. If X corresponds to a repeated root of the ϕ_1 cubic, the statement is not always true.

Except when all the roots of the ϕ_1 cubic are equal there is always *some* line such that if E be any motor coaxial with the line, $\phi\mathrm{E}$ is also coaxial with the line. [When all the roots of the ϕ_1 cubic are equal the statement is sometimes true and sometimes untrue.]

Let, now, ϕ stand for a real commutative self-conjugate. It can then always be put in the form—

$$\phi\mathrm{E} = -\mathrm{X}i\mathrm{SE}i-\mathrm{X}'j\mathrm{SE}j-\mathrm{X}''k\mathrm{SE}k,$$

where X, X′, X″ are three real scalar octonions, and i, j, k are three mutually perpendicular unit intersecting rotors. The cubic is $(\phi-\mathrm{X})(\phi-\mathrm{X}')(\phi-\mathrm{X}'') = 0$, so that it always has three real roots. These may be taken as X, X′, X″, even when it has an infinite number of roots. In this last case the X, X′, X″ of the equation $\phi\mathrm{E} = -\mathrm{X}i\mathrm{SE}i-\ldots$ have definite values which are called the *principal* roots of the cubic. Thus, there are always three mutually perpendicular intersecting lines such that if E be a motor coaxial with any one of them, $\phi\mathrm{E}$ is coaxial with E.

If two, but not three, of the roots of the ϕ_1 cubic are equal, and the two corresponding principal roots of the ϕ cubic are unequal, ϕ cannot be put in the form—

$$\phi\mathrm{E} = \mathrm{MAEB}+\mathrm{YE},$$

where A and B are constant motors and Y a constant scalar octonion. In all other cases ϕ can be put in this form, and A, B, and Y. are real. The cubic is

$$\{(\phi-\mathrm{Y})^2-\mathrm{A}^2\mathrm{B}^2\}\{(\phi-\mathrm{Y})+\mathrm{SAB}\} = 0.$$

When neither A nor B is a lator and they are not parallel, the principal axes (the three lines mentioned just now), of ϕ are the shortest distance of A and B, and the two lines which bisect this shortest distance and also bisect the angles between A and B.

A scalar octonion may be of any one of five types; one type is zero and the other four may be called positive and negative, scalar octonions and positive and negative scalar convertors. If A and B are two motors (not parallel and neither a mere lator), such that $\mathrm{SA}\phi\mathrm{B}$, $= 0$, A and B are said to be fully conjugate with regard to ϕ. If A, B, C are three such motors of which each pair is fully conjugate the number of the scalar octonions $\mathrm{SA}\phi\mathrm{A}$, $\mathrm{SB}\phi\mathrm{B}$, $\mathrm{SC}\phi\mathrm{C}$, of any type is the same as the number of the principal roots of the same

type. Making this fact the basis of a classification, it follows that there are (rejecting the case $\phi = 0$), thirty-four different types of ϕ.

If ψ is a commutative self-conjugate, such that $S_1 E \psi E$ is negative, and not zero for all motor values (except lators) of E, there are always three (and generally only three) real lines which are not all parallel to one plane such that any three real motors with these lines for axes form a fully conjugate set both with regard to ψ and ϕ.

A commutative self-conjugate is never a complete energy function, and can only be a partial energy function when it degenerates into a lator function.

The properties of the general function are examined by the help of "Grassmann's Ausdehungslehre." Motors are in Grassmann's own geometrical interpretations quantities of the second order. "Inner Multiplication," and the thory of "Normals," have in this case interpretations which depend on an arbitrarily chosen origin. Hence Grassmann does not apply his theories concerning these two to motors; but motors *may* be treated as quantities of the first order, and if they are, and if Sir Robert Ball's meaning of "reciprocal" is identified with Grassmann's meaning of "normal" a real motor with negative pitch is a "simple imaginary" quantity of the first order, and the "numerical value" of a lator or a rotor is zero. Grassmann generally assumes his quantities to be real and always assumes that a quantity is zero when its numerical value is zero. Hence nearly all his *theorems* require modification in our case but his *methods* are generally with some extensions applicable.

"Combinatorial Multiplication" receives several applications. A particular scalar combinatornal product of six motors is much used and also a motor combinatornal product of five motors. The latter is the motor (with a certain definite tensor) reciprocal to the five motors of the product. By aid of these two products, the following results (the first of which is contained in Ball's "Screws") are established. To every complex of order n there is a complex of order $6-n$ reciprocal and no motor not belonging to the latter, is reciprocal to the former. If (n) be a complex of order n, and $(6-n)$ an independent complex of order $6-n$, then if $\overline{(6-n)}$ is the complex reciprocal to (n) and $\overline{(n)}$ the complex reciprocal to $(6-n)$, $\overline{(n)}$ and $\overline{(6-n)}$ are independent complexes.

What is called "combinatorial variation" is a species of variation of which Grassmann's "linear" and "circular" variations are special cases. Particular cases of combinatorial variation are these two, and also "hyperbolic" and "conjugate" variations, the last being with reference to a given general self-conjugate function.

Let ϖ be a given real general self-conjugate function. The motors E of a given complex (n) of order n, for which $sE\varpi E = 0$ from what Sir Robert Ball calls a complex of the $(n-1)$, the order and second

degree. Two motors, E and F, are said to be conjugate with reference to ϖ, when $sE\varpi F =_s 0$. In the language of "Screws" they are conjugate motors of the complex of the second degree. ϖE is what Ball calls the polar of E.

When $\varpi = 1$ conjugacy reduces to reciprocity, and every theorem relating to the former becomes one concerning reciprocal motors.

A real motor, E, is called a positive, negative, or zero norm, according as $sE\varpi E$ is -1, $+1$, or zero. When $\varpi = 1$ a positive norm has positive pitch, a negative norm negative pitch, and a zero norm zero or infinite pitch, *i.e*, it is a rotor or lator. (n) can always be expressed in an infinite number of ways as a complex of n real independent conjugate norms. The number of norms of any type (positive, negative, or zero) is definite, and the complex of zero norms is a definite complex.

If (n) be a given complex of order n, and (m) a given complex of order m included in (n), another complex $(n-m)$ of order $n-m$ can always be found with the following properties. (m) and $(n-m)$ together make up (n), so that $(n-m)$ is also included in (n), and (m) and $(n-m)$ are independent. (m) consists of the positive norms $A_1\ A_2 \ldots$, the negative norms $B_1\ B_2 \ldots$, and the zero norms $C_1\ C_2 \ldots\ D_1\ D_2 \ldots$. $(n-m)$ consists of the positive norms $A_1'\ A_2' \ldots$, the negative norms $B_1'\ B_2', \ldots$, and the zero norms $C_1'\ C_2' \ldots\ D_1'\ D_2' \ldots$. The number of the norms $D_1\ D_2 \ldots$ is the same as the number of the norms $D_1'\ D_2' \ldots$. All pairs of these norms are conjugate except the following pairs of zero norms $(D_1\ D_1')$, $(D_2\ D_2') \ldots$. These last are such that

$$sD_1\varpi D_1' = sD_2\varpi D_2' = \ldots = -1.$$

The pair of motors A_1'', B_1'', deduced from any such pair of exceptional zero norms D_1, D_1' by the equations

$$A_1'' = (D_1+D_1')/\sqrt{2},\ B_1'' = (D_1-D_1')/\sqrt{2},$$

are conjugate, and are respectively a positive and negative norm of (n). Thus (n) consists of the following set of conjugate norms: (1) positive, $A_1 \ldots$, $A_1' \ldots$, $A_1'' \ldots$, (2) negative, $B_1 \ldots$, $B_1' \ldots$, $B_1'' \ldots$, (3) zero, $C_1 \ldots$, $C_1' \ldots$. (n) and $(n-m)$ are called semi-conjugate complexes.

Denoting the complex of $A_1\ A_2 \ldots$ by (A), that of $A_1\ A_2 \ldots B_1\ B_2 \ldots$ by (AB), &c., it is proved that (A′B′CC′D) contains all the motors of (n) which are conjugate to (m) or (ABCD), and no others. (A′B′CC′D) is therefore called the conjugate of (m). The sum of the orders of (m) and its conjugate exceeds n by the order of (C), which is the complex common to (m) and the (definite) zero norm complex (CC′) of (n). The conjugate of the conjugate of (m) is not in general (m) itself.

When (n) and (m) and nothing else is given the statements of the following table are established.

The following complexes are determinate.	The following complexes are arbitrary to the extent mentioned.
(C)	
(CC′)	(C′) is any complex which with (C) makes up (CC′)
(CC′D)	
(CD)	(D) ,, ,, (C) ,, (CD)
(ABCD)	(AB) ,, ,, (CD) ,, (ABCD)
(ABCC′D)	
(A′B′CC′D)	(A′B′) ,, ,, (CC′D) ,, (A′B′CC′D)
(ABA′B′CC′D)	

Also (D′) is indeterminate to the extent implied by the statement:—(DD′) is any complex which contains (D), is conjugate to (ABA′B′), and with (ABA′B′CC′) makes up (n). The individual motors C_1, C_2.... may be taken as any independent motors forming the determinate complex (C), and the individual motors D_1, D_2.... may be taken as any independent motors which with C_1, C_2.... form the determinate complex (CD); similarly for C_1', C_2'.... ; but not similarly for D_1', D_2'.... .

The table is much simplified, (1) when ϖ is an energy function and (2) when $\varpi = 1$ and $n = 6$.

If ϖ is any self-conjugate and E is one of a set of six conjugate motors, such that $sE\varpi E = 0$, then $\varpi E = 0$. If ϖ is an energy function (partial) and E is *any* motor, such that $sE\varpi E = 0$, then $\varpi E = 0$.

ϖ (a general self-conjugate) can always be expressed in an infinite number of ways in the form $\varpi E = -\Sigma f F s E F$, where F is one of six real constant independent motors, and f a corresponding ordinary scalar, which may be zero. The six motors which are reciprocal to each set of five of the F's form a conjugate set. When none of the f's are zero ϖ^{-1} has a unique intelligible meaning. The F's form a conjugate set with regard to ϖ^{-1}.

When (n) and (m) are given the *numbers* of motors of the different types A, A′, &c., are all definite numbers characteristic of ϖ. There are thus when $n = 6$, two methods of classifying ϖ. The number of types of ϖ (rejecting the case $\varpi = 0$), when classified according as it affects motors in (m) only, is 2, 5, 9, 14, 20, or 27, when m has the value 1, 2, 3, 4, 5, or 6 respectively. The number of types of ϖ, when classified according as it affects motors in (m) and in a semi-conjugate complex ($6-m$), is 27, 77, 125, 145, 125, 77, or 27, when m has the value 0, 1, 2, 3, 4, 5, or 6 respectively.

(n) being as before, ϖ being a given general self-conjugate, and ψ

a given energy function, the following n real independent motors, forming a conjugate set with regard to ϖ, can always be found, and in general uniquely; (1) $A_1 \ldots . \alpha_1 \ldots .$, positive norms with regard to ϖ; (2) $B_1 \ldots . \beta_1 \ldots .$, negative norms with regard to ϖ; (3) $C_1 \ldots .$, zero norms with regard to ϖ. The numbers in the two groups $\alpha_1 \ldots .$, $\beta_1 \ldots .$ are the same. The motors $A_1 \ldots .$, $B_1 \ldots .$, $C_1 \ldots .$, $\alpha_1 + \beta_1 \ldots .$, $\alpha_1 - \beta_1 \ldots .$, form a conjugate set (not norms in general) with regard to ψ, and of these $\alpha_1 - \beta_1 \ldots .$, are zero norms with regard both to ϖ and ψ. When there are any α's and β's there are not n real independent motors forming a common conjugate system of ϖ and ψ. If ψ is a *complete* energy function there are no α's and β's. Several particular cases of the general theorem are examined, those of especial importance being when $\varpi = 1$.

If ϖ and ϖ' are two general self-conjugates such that $sE\varpi E = sE'\varpi E$ for every motor E of a given complex, then $sE\varpi F = sE\varpi' F$ where E and F are any two motors of the complex. In particular, any motors of the complex which form a set conjugate with regard to ϖ are also conjugate with regard to ϖ'.

If (n) is a given complex of order n and $(6-n)$ a given independent complex of order $6-n$, and if ϖ is given then ϖ' can be determined uniquely, so that $sE_1\varpi' E_1 = sE'\varpi E_1$, $\varpi' E_2 = 0$ where E_1 is any motor of (n) and E_2 any motor of $(6-n)$. Also, whatever motor value E have $\varpi' E$ belongs to the complex reciprocal to $(6-n)$.

If ϕ is a general (not necessarily self-conjugate) function, such that it reduces every motor of (n) to a motor of (n), and reduces every motor of $(6-n)$ to zero, ϕ satisfies an n-tic. If the roots of this equation are a repeated r times, b repeated s times, &c., where $a, b, \ldots .$ are all different, there are certain definite independent complexes (r), $(s), \ldots .$ included in and making up (n), of orders $(r), (s), \ldots .$ corresponding to these roots. In (r) is included a complex (p), (not always definite) of order p, where p is any positive integer not greater than r, such that $(\phi - a)^p A = 0$ for every motor A of (p).

If p be *any* positive integer and e be a scalar different from both a and b, $(\phi - e)^p E$ is not zero, and belongs to the complex consisting of (r) and (s) if E itself belongs to that complex. A similar statement is true of the complex consisting of any number of the complexes (r), $(s), \ldots .$

If $(\phi - e)^p E = 0$ (p any positive integer) for every motor E of a complex of order q included in (n), e is a root of the n-tic repeated at least q times, and the complex is included in the complex corresponding to this root.

$(\phi - b)^s (\phi - c)^t \ldots . E$ where all the roots occur their full number of times, except a which is absent altogether, and where E is a motor of (n) belongs to the complex (r), and by giving a suitable value to E it may be made any motor of (r).

If $(\bar{n})$ is the reciprocal of $(6-n)$ and $(\overline{6-n})$ is the reciprocal of (n), then $(\bar{n})$ and $(\overline{6-n})$ stand towards ϕ', the conjugate of ϕ, in exactly the same way as (n) and $(6-n)$ stand towards ϕ.

The ϕ' n-tic is the same as the ϕ n-tic, and the complex $(\bar{r})$ corresponding to the root a of the ϕ' n-tic is reciprocal to (s), (t),.... and to $(6-n)$.

If ϕ be self-conjugate ($\phi' = \phi$) the complexes corresponding to the different roots are both conjugate, with regard to ϕ, and reciprocal.

When not one of the roots of the ϕ n-tic is zero ϕ^{-1} applied to motors of (n), and supposed to reduce them to motors of (n), has a unique intelligible meaning. It is, like ϕ, a general function. The roots of its n-tic are the reciprocals of the roots of the ϕ n-tic. The complex corresponding to any root of the ϕ^{-1} n-tic is the same as the complex corresponding to the corresponding root of the ϕ n-tic. ϕ'^{-1} is the conjugate of ϕ^{-1}.

If ϕ is real the coefficients of the n-tic are real, but not necessarily the roots. If a and b are two corresponding imaginary roots (*i.e.*, ab and $a+b$ are real), $r = s$. The complexes (r) and (s) are then imaginary, but the complex consisting of (r) and (s) is real. In this real complex is included a real complex $(2p)$ of order $2p$, where p is any positive integer not greater than r, such that $\{(\phi-a)(\phi-b)\}^p$ reduces every motor of $(2p)$ to zero.

A real general self-conjugate (octonion) ϖ differs from a quaternion self-conjugate, and differs from a Grassmann self-conjugate in that (1) the roots of its sextic ($n = 6$) may be imaginary, and (2) there may be not more than one motor, A, in the complex corresponding to a repeated root a, for which $(\varpi-a)\text{A} = 0$, whether a be real or imaginary. The roots of an energy function sextic are always real, and when for such a function a is not zero, every motor A of (r) is such that $(\phi-a)\text{A} = 0$. But when $a = 0$ there is not even in this case necessarily more than one motor A for which $\phi\text{A} = 0$. For a complete energy function a is never zero.

If ϕ degenerates into a commutative function the roots of its sextic are the roots of what was above called the ϕ_1 cubic each repeated twice.

In particular, if ϕ degenerates into a self-conjugate commutative function the roots of the sextic are all real, and consist of three pairs of equal roots. The three corresponding complexes are three sets of coaxial motors, whose axes are three mutually perpendicular intersecting lines. Except when what were called the principal roots of the ϕ cubic are all ordinary scalars, there are not in this case six co-reciprocal motors, which also form a conjugate set.

Combinatorial variation is applied to prove some other facts. ϖ being a general self-conjugate, and (n), as before, the sum of the reciprocals of the pitches of $\text{SF}_1\varpi\text{F}_1$, $\text{SF}_2\varpi\text{F}_2$, ... where F_1, F_2 ....

are a conjugate set of n independent motors of (n) is constant. In particular, putting $\varpi = 1$ we get the theorem proved in "Screws," that the sum of the reciprocals of n independent co-reciprocal motors of (n) is constant. If A, B, C are any three independent motors of a complex of the third order the pitch of SABC is constant, and if A, B are any two independent motors of a complex of the second order the pitch of MAB is constant. Or adopting the rotation above the sum of $d \cot \theta - e \tan \phi$, and the pitches of A, B, C is constant in the first case, and the sum of $d \cot \theta$ and the pitches of A and B is constant in the second case.

The most general forms of complexes of all orders, expressed whenever possible as consisting of reciprocal motors with axes along mutually perpendicular intersecting lines, are given; as also the reciprocal complexes in a similar form.

The differentiation of octonion functions is considered.

It is shown that physical problems may be treated in a manner which, so far as appears from the present trials is in most cases practically identical with quaternion method. A symbolic *rotor* ∇ is defined, which has properties very similar to the *vector* ∇ of quaternions. Integration theorems corresponding to the well known quaternion ones are given. The octonion treatment of strain and of intensities and fluxes is given at some length. The octonion formulæ, though bearing a somewhat more extended meaning than the quaternion formulæ, are surprisingly similar in form to the latter. On the whole, from this part of the paper, it cannot be said that octonions prove more efficient than quaternions, though what would appear were the subject more developed cannot at present be said.

The following are some of the applications made in the last division of the paper:—

A twist means a general displacement of a rigid body. A twist can always be effected in an infinite number of ways by two rotations. The following construction suffices to find any two such rotations. Take any line 1 intersecting the axis of the twist perpendicularly. Let 1 become 2 when it is subjected to half the given twist. Take any transversal 3 of 1 and 2. Then double the rotation that converts 1 into 3, followed by double the rotation that converts 3 into 2, will effect the given twist. A right-about-turn means a rotation through two right angles; thus it is completely specified by its axis. A twist can always be effected in an infinite number of ways by two right-about-turns. The following suffices to obtain the axes in all cases. The axis of the first right-about-turn is any line intersecting the axis of the twist perpendicularly. The second axis is obtained from the first by giving to the latter half the given twist. As is well known, two equal parallel and opposite rotations combine into a translation. The translation is compounded of two translations, the

one being equal and parallel to the perpendicular from the first axis on the second, and the other equal and parallel to the same perpendicular when it has been first rotated with the second rotation and then reversed. To combine two twists, take two lines, 1 and 2, such that half the first twist brings 1 into coincidence with the shortest distance between the axes, and half the second twist brings the shortest distance into coincidence with two. Then the axis of the resultant twist is the shortest distance between 1 and 2 and the twist itself double the twist about this axis, which will bring 1 into coincidence with 2.

The geometrical properties of the second order and third order complexes, as given by Sir Robert Ball in "Screws," are established.

The octonion treatment of the motion of a single rigid body is considered at some length. The treatment is very analogous in many parts to the quaterion treatment of the motion of a rigid body with one point fixed. The variation in time of the pitch and position of the velocity motor of a rigid body subject to no external forces is considered.

This leads to the consideration of a curve—called the "normal cone curve"—related to the polhode. The polhode lies on a quadric cone whose vertex is the centre of the Poinsot ellipsoid. The normal cone curve lies on the quadric cone whose vertex is the same point, and which is normal to the polhode cone. Defining a polhode as the locus of points on a quadric at which the tangent planes touch a concentric sphere, it is shown that a real polhode on a real quadric is always a polhode on a second real coaxial quadric, and that the normal cone curve is a polhode on each of two other coaxial quadrics which are both real or both imaginary. The original polhode is similar and similarly situated to the normal cone curve of the original normal cone curve when the latter is regarded as a polhode. Several reciprocal relations between the two pairs of quadrics are established. If either of the original polhode quadrics is an hyperboloid of two sheets, the normal cone curve quadrics are imaginary, though the normal cone curve itself is always real. When the four quadrics are all real, not more than three of them can be ellipsoids, though they may all be hyperboloids of one sheet. A figure drawn to scale is given in which each pair consists of an ellipsoid and an hyperboloid of one sheet, and in which each polhode is identical with the normal cone curve of the other. The two polhode quadrics coalesce if, and only if, one is a sphere. The general solution given breaks down in certain limiting cases. These cases are examined, and all prove to have simple geometrical properties. The methods adopted in these particular operations are to all intents and purposes quaternion methods, and they illustrate, what appears frequently throughout the

paper, that quaternion methods may be regarded practically as a particular case of octonion methods.

When the motion of the rigid body is always in the neighbourhood of an absolutely stable position we get the theory of "Screws." The more general theorems in connection with this are given in a somewhat different form from Sir Robert Ball's. Thus, for instance, we have a "generalised force motor," which in a particular case becomes the "reduced wrench." It is pointed out that the latter is not always intelligible. Some form of the former is always intelligible, as also is a particular form of it called the "virtual force motor." The corresponding "virtual impulse motor" is the impulse motor due to the external impulses and the reactions of the constraints, but a similar statement does not hold for the virtual force motor

IV. "On the Formation and Structure of Dental Enamel." By J. LEON WILLIAMS, D.D.S., L.D.S. Communicated by Professor SCHÄFER, F.R.S. Received December 4, 1894.

(Abstract.)

The special points in the formation and structure of enamel which I have attempted to elucidate in this paper may be summarised as follows:—

1st. The existence of a very thin membrane, or a structure of membrane-like appearance, lying between the ameloblasts and the forming enamel, and also between these cells and those of the stratum intermedium. I have also, in many specimens, seen a similar membrane covering the odontoblasts.

2nd. The formation of enamel by deposit and not by cell calcification. This deposit probably consists of two distinct cell products—a granular plasm and spherules of calcoglobulin.

3rd. The relation of the cells of the stratum intermedium to true secreting tissue; this relation being especially marked in the enamel organs of the rat and mouse.

4th. An intricate vascular network in the stratum intermedium. I should also mention that I have seen a free distribution of blood vessels in the odontoblastic layer of cells in the mouse, rat, and calf, as well as in human embryos, thus conclusively proving that these cells are not calcified.

5th. The fibrous character of enamel in many of the lower animals, and the change of these fibres into more or less regularly arranged granules in the monkey and in man.

6th. That the varicosities of the enamel rods are not caused by acids (although often rendered more clear to view by acid treatment)